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DISPERSION MODELS AND ESTIMATION OF DISPERSION EFFECTS

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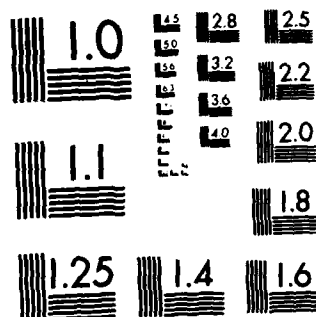
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Effects In Replicated Factorial Experiments

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Dispersion Models and Estimation of Dispersion Effects In  
Replicated Factorial Experiments

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Summary

In this paper, we consider the problem of estimation of dispersion effects of factors in replicated factorial experiments under a general dispersion model. We also characterize the arrays so that the estimation of dispersion effects is possible. The problem considered in this paper arises in quality control studies and the methodologies are applicable to industrial experiments.

Key Words: Adjusted residuals, Arrays, Dispersion effects, Factorial experiments, Linear models, Quality control.

Short Running Title: Dispersion Models and Estimation of Dispersion Effects

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# 1. Introduction

We consider a  $2^m$  factorial experiment under a completely randomized design. The runs are denoted by  $\underline{t}_i' = (t_{i1}, \dots, t_{iu}, \dots, t_{im})$  where  $t_{iu}$  is the level of the factor  $u$  in the run  $i$ ,  $t_{iu} = 0, 1$ ;  $u = 1, \dots, m$ ;  $i = 1, \dots, n$ . Let  $T$  ( $n \times m$ ) be a matrix whose  $i$ th row is  $\underline{t}_i'$ ,  $i = 1, \dots, n$ . The matrix  $T$  is called the array (or the design) for the experiment. We take  $r_i (> 1)$  observations for the  $i$ th ( $i = 1, \dots, n$ ) run of the experiment. The experiment just described is very common in quality control studies (see Taguchi and Wu 1985) where a major goal is to evaluate the sensitivity of the manufacturing process to noise. The runs  $\underline{t}_i'$ ,  $i = 1, \dots, n$ , in  $T$  are called the level combinations of  $m$  control factors. An important problem is to find an optimum combination of levels of control factors in view of reducing the process variability due to noise factors. A list of noise factors likely to affect the process variability is first made. Various level combinations of noise factors that provide a good representation of noise are then selected. For the  $i$ th run  $\underline{t}_i'$ , we collect  $r_i$  observations corresponding to  $r_i$  level combinations of noise factors. The  $r_i$  observations for a run in  $T$  are called  $r_i$  replicated observations. The variability in the  $r_i$  replicated observations for the  $i$ th run  $\underline{t}_i'$  is attributed to the process variability due to noise in the experiment.

Let  $y_{ij}$  be the  $j$ th observation on the  $i$ th run  $\underline{t}_i'$  in  $T$ ,  $j = 1, \dots, r_i$ ,  $i = 1, \dots, n$ . We denote  $\underline{y} = (y_{11}, \dots, y_{1r_1}; \dots; y_{n1}, \dots, y_{nr_n})'$ . There are  $N = r_1 + \dots + r_n$  observations in  $\underline{y}$ . The model for the experiment is

$$E(\underline{y}) = X\underline{\beta}, \quad (1)$$

$$V(\underline{y}) = \Sigma, \quad (2)$$

where  $X(N \times p)$  is a known matrix that depends on the array  $T$ ,  $\underline{\beta}(p \times 1)$  is a vector of factorial effects considered in the experiment. The matrix  $\Sigma$  is an unknown diagonal matrix where the diagonal elements for all observations on  $\underline{t}_i' = (t_{i1}, \dots, t_{im})$  are equal to  $\sigma^2(t_{i1}, \dots, t_{im}) = \sigma^2(\underline{t}_i')$ . The parameters  $\sigma^2(t_{i1}, \dots, t_{iu} = 1, \dots, t_{im})$  are called the dispersions of the factor  $u(u = 1, \dots, m)$  at level 1. The parameters  $\sigma^2(t_{i1}, \dots, t_{iu} = 0, \dots, t_{im})$  are called the dispersions of the factor  $u(u = 1, \dots, m)$  at level 0. We are interested in comparing the dispersions at level 1 with the dispersions at level 0 for every factor. For this purpose, we define the dispersion effect of the factor  $u(u = 1, \dots, m)$  as

$$\Sigma c(t_{i1}, \dots, t_{iu} = 1, \dots, t_{im}) \sigma^2(t_{i1}, \dots, t_{iu} = 1, \dots, t_{im}) - \Sigma d(t_{i1}, \dots, t_{iu} = 0, \dots, t_{im}) \sigma^2(t_{i1}, \dots, t_{iu} = 0, \dots, t_{im}), \quad (3)$$

where  $c(t_{i1}, \dots, t_{iu} = 1, \dots, t_{im})$  and  $d(t_{i1}, \dots, t_{iu} = 0, \dots, t_{im})$  are known constants with  $\Sigma c(t_{i1}, \dots, t_{iu} = 1, \dots, t_{im}) = \Sigma d(t_{i1}, \dots, t_{iu} = 0, \dots, t_{im}) = 1$ . The dispersion effect is a contrast of the dispersions at level 1 and level 0. If the dispersion effect is greater than 0, we then prefer level 0 over level 1 of the factor. If the dispersion effect is less than 0, we then prefer level 1 over level 0. If the dispersion effect is zero then levels 1 and 0 are equally preferable. Notice that the preference of one level over the other level depends on the dispersion effect considered or in other words, on the values of  $c$ 's and  $d$ 's. The linear functions of dispersion effects of the form (3) are used in measuring how the dispersion effect of one factor varies at different levels of the other factors [see Box and Meyer (1986), Srivastava

(1987)]. In this paper we consider the problem of estimating meaningful dispersion effects of the factors. We also characterize arrays so that the estimation of meaningful dispersion effects is possible. Although in (3) we use the contrast of level 1 and level 0 dispersions for the dispersion effect, one may also take the ratio of two dispersions considered (see Box and Meyer (1986)). The use of any of these definitions will however give the same conclusion since each of them compares the dispersions at level 1 and level 0. We assume throughout the paper that for the fitted model to the data there is no significant lack of fit.

The dispersion model and dispersion effects are considered in the work of Taguchi (see Taguchi and Wu 1985), Box (see Box and Meyer 1986), Kacker and Shoemaker (1986), Phadke (1986), Ghosh (1987), Srivastava (1987), Nair and Pregibon (1988). This paper makes a further contribution to this area of research.

## 2. Estimation of Dispersion Effects.

We choose a factor out of  $m$  factors and present estimators of some meaningful dispersion effects of the chosen factor. For clarity, we do not introduce any notation for the chosen factor. We define the following indicator variable for  $i = 1, \dots, n$ , so that it is possible to identify the runs at which the factor appears at level 1 and level 0, respectively:

$$\delta_i = \begin{cases} 1 & \text{if the level of the factor in the } i\text{th run is 1,} \\ 0 & \text{if the level of the factor in the } i\text{th run is 0.} \end{cases}$$

A meaningful dispersion effect of the form (3) is

$$\sigma^2(1) - \sigma^2(0) = \frac{\sum_{i=1}^n \delta_i \sigma^2(\underline{t}_i)}{\sum_{i=1}^n \delta_i} - \frac{\sum_{i=1}^n (1-\delta_i) \sigma^2(\underline{t}_i)}{\sum_{i=1}^n (1-\delta_i)}. \quad (4)$$

The dispersion effect (4) is the difference between the average dispersions at level 1 and level 0 of the chosen factor. Let  $\bar{y}_i$  be the mean of all observations on the  $i$ th run,  $i = 1, \dots, n$ . The sum of squares of pure error is

$$SSPE = \sum_{i=1}^n \sum_{j=1}^{r_i} (y_{ij} - \bar{y}_i)^2 = \sum_{i=1}^n \sum_{j=1}^{r_i} \delta_i (y_{ij} - \bar{y}_i)^2 + \sum_{i=1}^n \sum_{j=1}^{r_i} (1-\delta_i) (y_{ij} - \bar{y}_i)^2.$$

We now write

$$S_1^2(1) = \frac{\sum_{i=1}^n \sum_{j=1}^{r_i} \delta_i (y_{ij} - \bar{y}_i)^2}{\sum_{i=1}^n \delta_i (r_i - 1)},$$

$$S_1^2(0) = \frac{\sum_{i=1}^n \sum_{j=1}^{r_i} (1-\delta_i) (y_{ij} - \bar{y}_i)^2}{\sum_{i=1}^n (1-\delta_i) (r_i - 1)}. \quad (5)$$

It can be seen that  $E(S_1^2(1) - S_1^2(0)) = \sigma^2(1) - \sigma^2(0)$ . Thus  $S_1^2(1) - S_1^2(0)$  is an unbiased estimator of  $\sigma^2(1) - \sigma^2(0)$ .

It is important to estimate the dispersion effect which is a simple contrast of dispersions at levels 1 and 0, or in other words, to estimate  $\sigma^2(t_1, \dots, t_{u-1}, 1, t_{u+1}, \dots, t_m) - \sigma^2(t_1, \dots, t_{u-1}, 0, t_{u+1}, \dots, t_m)$  for a  $(t_1, \dots, t_{u-1}, t_{u+1}, \dots, t_u)$ . This is possible if both runs  $(t_1, \dots, t_{u-1}, 1, t_{u+1}, \dots, t_m)$  and  $(t_1, \dots, t_{u-1}, 0, t_{u+1}, \dots, t_m)$  appear as rows in  $T$ . Suppose that both runs are present in  $T$  and they appear in the rows  $i_1$

$i_2$  of  $T$ . The unbiased estimator of the parameter of interest  $\sigma^2(\underline{t}'_{i_1}) - \sigma^2(\underline{t}'_{i_2})$  is

$$\frac{\sum_{j=1}^{r_{i_1}} (y_{i_1 j} - \bar{y}_{i_1})^2}{(r_{i_1} - 1)} - \frac{\sum_{j=1}^{r_{i_2}} (y_{i_2 j} - \bar{y}_{i_2})^2}{(r_{i_2} - 1)}. \quad (6)$$

Suppose that the runs  $(t'_1, \dots, t'_{u-1}, 1, t'_{u+1}, \dots, t'_m)$  and  $(t'_1, \dots, t'_{u-1}, 0, t'_{u+1}, \dots, t'_m)$  also appear in the rows  $i_3$  and  $i_4$  of  $T$ . We then unbiasedly estimate the dispersion effect  $\sigma^2(\underline{t}'_{i_3}) - \sigma^2(\underline{t}'_{i_4})$ . The linear functions of the dispersion effects  $\frac{1}{2} [\sigma^2(\underline{t}'_{i_1}) - \sigma^2(\underline{t}'_{i_2})] + \frac{1}{2} [\sigma^2(\underline{t}'_{i_3}) - \sigma^2(\underline{t}'_{i_4})]$  and  $\frac{1}{2} [\sigma^2(\underline{t}'_{i_1}) - \sigma^2(\underline{t}'_{i_2})] - \frac{1}{2} [\sigma^2(\underline{t}'_{i_3}) - \sigma^2(\underline{t}'_{i_4})]$  are very important in terms of the "main effects" and the "interactions" considered in Box and Meyer (1986), Srivastava (1987).

Note that the first linear function is in the form (3) and is thus again a dispersion effect of the factor chosen. The second linear function is not of the form (3) but a contrast of two dispersion effects of the form (3). There is a possibility for an array  $T$  that the unbiased estimation of  $\sigma^2(\underline{t}'_{i_1}) - \sigma^2(\underline{t}'_{i_2})$  is not at all possible for any  $(t_1, \dots, t_{u-1}, t_{u+1}, \dots, t_m)$  but the unbiased estimation of  $\sigma^2(1) - \sigma^2(0)$  is still possible. This however is not a desirable situation and the corresponding array  $T$  is not well chosen. In section 3 we discuss this issue in detail.

We denote  $\underline{y}_u$  as the vector of observations corresponding to the runs with the chosen factor at level  $u$ ,  $u = 1, 0$ . Notice that  $\underline{y}$  consists of  $\underline{y}_1$  and  $\underline{y}_0$ . Let  $X_u$  be the submatrix of  $X$  corresponding to  $\underline{y}_u$ ,  $\Sigma_u$  be the submatrix of  $\Sigma$  corresponding to  $\underline{y}_u$ ,  $\hat{\underline{y}}_u$  be the ordinary least squares fitted values for  $\underline{y}_u$ . We have  $\hat{\underline{y}}_u = X_u(X'X)^{-1}X'\underline{y}$ ,  $u = 1, 0$ . We denote

$\underline{y}_u - \hat{\underline{y}}_u = \underline{r}_u \underline{y}$ , Rank  $\underline{r}_u = V_u$  and  $V_u S_2^2(u) = \underline{y}' \underline{r}_u' \underline{r}_u \underline{y}$ ,  $u = 1, 0$ . Thus  $V_u S_2^2(u)$  is the sum of squares of the ordinary least squares residuals at level  $u$  of the chosen factor.

**Theorem 1.** We have for  $u = 1, 0$ ,

- a.  $E(V_u S_2^2(u)) = \text{Trace} (\underline{r}_u \underline{r}_u' \Sigma_u \underline{r}_u \underline{r}_u' + \underline{r}_u \underline{r}_{(1-u)}' \Sigma_{(1-u)} \underline{r}_{(1-u)} \underline{r}_u')$ ,
- b.  $E(V_u S_2^2(u)) = \text{Trace} \underline{r}_u \underline{r}_u' \Sigma_u \underline{r}_u \underline{r}_u'$ , if and only if  $\underline{r}_u \underline{r}_{(1-u)}' = 0$ , i.e.,  
 $\underline{r}_1 \underline{r}_0' = 0$ .

**Proof.** Since we have assumed no significant lack of fit for the fitted model, we get for  $u = 1, 0$ ,  $E(\underline{y}' \underline{r}_u' \underline{r}_u \underline{y}) = \text{Trace} \underline{r}_u' \underline{r}_u \Sigma = \text{Trace} \underline{r}_u \Sigma \underline{r}_u'$ . It can be seen that the submatrix of  $\underline{r}_u$  corresponding to  $\underline{y}_u$  and  $\underline{y}_{(1-u)}$  are  $\underline{r}_u \underline{r}_u'$  and  $\underline{r}_u \underline{r}_{(1-u)}'$ . Thus we have  $\underline{r}_u \Sigma \underline{r}_u' = \underline{r}_u \underline{r}_u' \Sigma_u \underline{r}_u \underline{r}_u' + \underline{r}_u \underline{r}_{(1-u)}' \Sigma_{(1-u)} \underline{r}_{(1-u)} \underline{r}_u'$ . Hence the part a of Theorem 1 is true. Since  $\text{Trace} \underline{r}_u \underline{r}_{(1-u)}' \Sigma_{(1-u)} \underline{r}_{(1-u)} \underline{r}_u'$  is the sum of squares of all elements in  $\underline{r}_u \underline{r}_{(1-u)}' \Sigma_{(1-u)}^{1/2}$  and furthermore  $\Sigma_{(1-u)}^{1/2}$  is a diagonal matrix with all diagonal elements positive, the part b of Theorem 1 is true. This completes the proof of the theorem.

It follows from Theorem 1 that in case  $\underline{r}_1 \underline{r}_0' = 0$ , the unbiased estimator of the dispersion effect

$$\frac{\text{Trace} \underline{r}_1 \underline{r}_1' \Sigma_1 \underline{r}_1 \underline{r}_1'}{\text{Trace} \underline{r}_1 \underline{r}_1'} - \frac{\text{Trace} \underline{r}_0 \underline{r}_0' \Sigma_0 \underline{r}_0 \underline{r}_0'}{\text{Trace} \underline{r}_0 \underline{r}_0'}, \quad (7)$$

is  $(V_1 S_2^2(1) / \text{Trace} \underline{r}_1 \underline{r}_1') - (V_0 S_2^2(0) / \text{Trace} \underline{r}_0 \underline{r}_0')$ . The dispersion effect (7) is of the form (3).

When  $\underline{r}_1 \underline{r}_0' \neq 0$ , two vectors of residuals  $\underline{r}_1 \underline{y}$  and  $\underline{r}_0 \underline{y}$  at levels 1 and 0 of the factor are correlated under the model (1-2). We present a vector  $\underline{r}_{(1-u)} \underline{y}$  of "adjusted residuals" at level  $(1-u)$  of the factor,

adjusted w.r.t.  $r_{u\bar{y}}$  ( $u = 1, 0$ ) so that  $r_{0a\bar{y}}$  and  $r_{1a\bar{y}}$  are uncorrelated under (1-2). Let  $r_{u1}(V_u \times N)$  be a submatrix of  $r_u$  so that  $\text{Rank } r_{u1} = V_u$ ,  $u = 1, 0$ . We write for  $u = 1, 0$ ,

$$r_{(1-u)a} = r_{(1-u)}(I - r'_{u1}(r_{u1}r'_{u1})^{-1}r_{u1}). \quad (8)$$

It can be seen that  $r_{u1}r'_{(1-u)a} = 0$  and hence  $r_u r'_{(1-u)a} = 0$ . It can be checked that  $\text{Rank } r_{(1-u)a} = ((N-p) - V_u) = V_{(1-u)a}$  (say). We have for  $u = 1, 0$ ,

$$r_{(1-u)a\bar{y}} = r_{(1-u)}(I - r'_{u1}(r_{u1}r'_{u1})^{-1}r_{u1}) r'_{(1-u)\bar{y}}. \quad (9)$$

Thus for  $u = 1, 0$ ,  $r_{(1-u)a\bar{y}}$  depends on  $\bar{y}$  only through  $\bar{y}_{(1-u)}$  and, moreover,  $\text{Cov}(r_{0a\bar{y}}, r_{1a\bar{y}}) = 0$ , i.e., they are uncorrelated under (1-2). Let  $r_{(1-u)a1}$  be a submatrix of  $r_{(1-u)}(I - r'_{u1}(r_{u1}r'_{u1})^{-1}r_{u1})r'_{(1-u)}$  with rank  $V_{(1-u)a}$ ,  $u = 1, 0$ . We now have the sum of squares of the set of linear functions  $r_{ual}\bar{y}_u$  [see Scheffé 1959] as

$$SS(r_{ual}\bar{y}_u) = \bar{y}'_u r'_{ual} [r_{ual}r'_{ual}]^{-1} r_{ual}\bar{y}_u, \quad (10)$$

with d.f.  $V_{ua}$ ,  $u = 1, 0$ . We define for  $u = 1, 0$ ,

$$S^2_{3a}(u) = [SS(r_{ual}\bar{y}_u)/V_{ua}]. \quad (11)$$

It can be seen that for  $u = 1, 0$ ,

$$E[V_{ua} S^2_{3a}(u)] = \text{Trace}(r_{ual}r'_{ual})^{-1} r_{ual} \Sigma_u r'_{ual}. \quad (12)$$

Thus the unbiased estimator of the dispersion effect

$$\frac{\text{Trace}(r_{1a1}r'_{1a1})^{-1} r_{1a1} \Sigma_1 r'_{1a1}}{V_{1a}} - \frac{\text{Trace}(r_{0a1}r'_{0a1})^{-1} r_{0a1} \Sigma_0 r'_{0a1}}{V_{0a}}, \quad (13)$$

is  $S^2_{3a}(1) - S^2_{3a}(0)$ . Again, the dispersion effect (13) is of the form (3).

### 3. Characterization of Arrays.

In this section we observe that for some arrays it may not be possible to estimate unbiasedly both  $\sigma^2(t_{i1}, \dots, t_{iu} = 1, \dots, t_{im})$  and  $\sigma^2(t_{i1}, \dots, t_{iu} = 0, \dots, t_{im})$  for any  $(t_{i1}, \dots, t_{i(u-1)}, t_{i(u+1)}, \dots, t_{im})$ . We present the characterizations of the arrays so that the unbiased estimation of both  $\sigma^2(t_{i1}, \dots, t_{iu} = 1, \dots, t_{im})$  and  $\sigma^2(t_{i1}, \dots, t_{iu} = 0, \dots, t_{im})$  are possible for at least one  $(t_{i1}, \dots, t_{i(u-1)}, t_{i(u+1)}, \dots, t_{im})$  and for every  $u$  in  $\{1, \dots, m\}$ .

#### 3.1. Orthogonal Arrays.

We first consider the orthogonal array  $T$  ( $n \times m$ ) whose rows  $\underline{t}_i'$ ,  $i = 1, \dots, n$ , satisfy

$$A \underline{t}_i = \underline{c} \text{ over GF}(2), \quad (14)$$

where  $A$  ( $s \times m$ ) is a known matrix of rank  $s$  and with two distinct elements 0 and 1,  $\underline{c}$  ( $s \times 1$ ) is a known vector with at most two distinct elements 0 and 1,  $\text{GF}(2)$  is the Galois Field with two elements 0 and 1. We have  $n = 2^{m-s}$ . Such an orthogonal array is called a type-A orthogonal array (see Srivastava and Chopra 1973). We write the matrix  $A$  in (14) as  $A = [\underline{a}_1, \underline{a}_2, \dots, \underline{a}_m]$ , where  $\underline{a}_u$  ( $s \times 1$ ) is the  $u$ th ( $u = 1, \dots, m$ ) column of  $A$ .

**Theorem 2.** Consider a run  $\underline{t}' = (t_1, \dots, t_{u-1}, t_u, t_{u+1}, \dots, t_m)$  of the type-A orthogonal array  $T$ . Then  $\sigma^2(t_1, \dots, t_{u-1}, t_u = 1, t_{u+1}, \dots, t_m)$  and  $\sigma^2(t_1, \dots, t_{u-1}, t_u = 0, t_{u+1}, \dots, t_m)$  are both unbiasedly estimable if and only if the  $u$ th column of  $A$ , i.e.,  $\underline{a}_u$  is a null vector.

**Proof.** We denote  $\underline{t}'_{(1)} = (t_1, \dots, t_{u-1}, t_u = 1, t_{u+1}, \dots, t_m)$  and  $\underline{t}'_{(0)} = (t_1, \dots, t_{u-1}, t_u = 0, t_{u+1}, \dots, t_m)$ . We have  $\underline{t}'_{(1)} + \underline{t}'_{(0)} = \underline{e}_u$  over  $\text{GF}(2)$ , where  $\underline{e}_u$  is the vector with the  $u$ th element unity and the other elements are zero. We can estimate both  $\sigma^2(\underline{t}'_{(1)})$  and  $\sigma^2(\underline{t}'_{(0)})$  if and only if both

$\underline{t}'_{(1)}$  and  $\underline{t}'_{(0)}$  appear as rows in  $T$ . (We know that one of them is already in  $T$ .) If both  $\underline{t}'_{(1)}$  and  $\underline{t}'_{(0)}$  are in  $T$  then we have  $A\underline{t}_{(1)} = A\underline{t}_{(0)} = \underline{c}$ . We get  $A\underline{t}_{(1)} + A\underline{t}_{(0)} = A\underline{e}_u$ , i.e.,  $\underline{c} + \underline{c} = \underline{a}_u$ . But,  $\underline{c} + \underline{c} = 2\underline{c} = \underline{0}$  over  $GF(2)$ . Thus the 'only if' part is true. The 'if' part is trivial. This completes the proof.

We want to estimate  $\sigma^2(\underline{t}'_{(1)})$  and  $\sigma^2(\underline{t}'_{(0)})$  for at least one  $(t_1, \dots, t_{u-1}, t_{u+1}, \dots, t_m)$  and for every factor. It follows from Theorem 2 that this is possible if and only if the matrix  $A$  in (14) is a null matrix. Then the type-A orthogonal array  $T$  does not exist. We therefore conclude that the type-A orthogonal arrays are not useful in the unbiased estimation of the dispersion effects of the type  $\sigma^2(\underline{t}'_{(1)}) - \sigma^2(\underline{t}'_{(0)})$  for every factor out of  $m$  factors. Although the type-A orthogonal arrays are widely used in quality control studies, we thus observe a serious drawback of such arrays.

We now consider the orthogonal array  $T$  ( $n \times m$ ) whose rows  $\underline{t}'_i$ ,  $i = 1, \dots, n$ , are solutions of

$$A\underline{t}_i = \underline{c}_k \text{ over } GF(2), k = 1, \dots, f \quad (15)$$

where  $A$  ( $s \times m$ ) and  $\underline{c}_k$  ( $s \times 1$ ) are known as in (14),  $f$  is also known. We have  $n = f 2^{m-s}$ . Such an orthogonal array is called a type-B orthogonal array (see Srivastava and Chopra 1973).

Example 1. Consider a  $2^5$  factorial experiment with 8 runs which are rows of the matrix  $T$  ( $8 \times 5$ ). We take

$$A\underline{x} = \begin{pmatrix} x_1 + x_2 \\ x_3 + x_4 \\ x_2 + x_4 + x_5 \end{pmatrix}. \quad (16)$$

Thus  $s = 3$ . We also take

$$\underline{c}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{c}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Thus  $f = 2$ . The matrix  $T$  is given below.

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}. \quad (17)$$

The matrix  $T$  in (17) is a type-B orthogonal array of strength 2. We have  $n = f 2^{m-s} = 2 \cdot 2^{5-3} = 8$ . Using this array we are able to estimate orthogonally all main effects under the assumption that 2-factor and higher order interactions are all zero (Resolution III or Main Effect Plan).

**Theorem 3.** Consider a run  $\underline{t}' = (t_1, \dots, t_{u-1}, t_u, t_{u+1}, \dots, t_m)$  of the type B orthogonal array  $T$  satisfying (15). Then  $\sigma^2(t_1, \dots, t_u = 1, \dots, t_m) = \sigma^2(\underline{t}'_{(1)})$  and  $\sigma^2(t_1, \dots, t_u = 0, \dots, t_m) = \sigma^2(\underline{t}'_{(0)})$  are both unbiasedly estimable if and only if the  $u$ th ( $u = 1, \dots, m$ ) column  $\underline{a}_u$  of  $A$  in (15) is  $\underline{a}_u = \underline{c}_k + \underline{c}_{k'}$  for  $k, k'$  in  $\{1, \dots, f\}$ .

**Proof.** We have  $\underline{t}_{(1)} + \underline{t}_{(0)} = \underline{e}_u$  over  $GF(2)$ . Thus  $A\underline{t}_{(1)} + A\underline{t}_{(0)} = A\underline{e}_u = \underline{a}_u$ . Now if  $\underline{t}_{(1)}$  and  $\underline{t}_{(0)}$  are in  $T$ , we take  $A\underline{t}_{(1)} = \underline{c}_k$  and  $A\underline{t}_{(2)} = \underline{c}_{k'}$ . Hence  $\underline{a}_u = \underline{c}_k + \underline{c}_{k'}$ . Suppose now that  $\underline{a}_u = \underline{c}_k + \underline{c}_{k'}$ . We have to show that both  $\underline{t}_{(1)}$  and  $\underline{t}_{(0)}$  are in  $T$ . We know that one of  $\underline{t}_{(1)}$  and  $\underline{t}_{(0)}$  is already in  $T$  depending whether the  $u$ th element  $t_u$  in  $\underline{t}$  is equal to 1 or

0. Assuming  $\underline{t}_{(1)}$  is in T, we show that  $\underline{t}_{(0)}$  is also in T. It can be seen that

$$A\underline{t}_{(0)} = A(\underline{e}_u - \underline{t}_{(1)}) = \underline{a}_u - \underline{c}_k = \underline{c}_{k'}.$$

Thus  $\underline{t}_{(0)}$  is also in T. The rest is clear. This completes that proof.

We observe that if a type-B orthogonal array satisfies the condition of Theorem 3, then the type-B orthogonal array whose runs are solutions of  $A\underline{t}_i = \underline{j} - \underline{c}_k$ ,  $k = 1, \dots, f$ ,  $i = 1, \dots, n$ ,  $\underline{j}' = (1, 1, \dots, 1)$ , also satisfies the condition of Theorem 3.

Example 2. We again consider the Example 1. The matrix A is given below.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Note that none of the columns in A can be expressed as the sum of  $\underline{c}_1$  and  $\underline{c}_2$ . Thus for the type-B orthogonal array T in (17), the condition in Theorem 3 does not hold. We now consider a  $2^5$  factorial experiment with 16 runs. We take  $A\underline{x}$  as in (16),  $f = 4$ ,  $\underline{c}_1$  and  $\underline{c}_2$  as in Example 1 and

$$\underline{c}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \underline{c}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The matrix T (16x5) consists of all runs in (17) and the following 8 additional runs.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The matrix  $T$  ( $16 \times 5$ ) is a type-B orthogonal array of strength 1 but not of strength 2. However, it is nearly an orthogonal array of strength 2. The condition of Theorem 3 holds for  $T$  ( $16 \times 5$ ). Hence  $\sigma^2(\underline{t}'_1)$  and  $\sigma^2(\underline{t}'_0)$  are both unbiasedly estimable for a  $(t_1, \dots, t_{u-1}, t_{u+1}, \dots, t_m)$  and for every chosen factor. We thus observe that we need more runs in the array to satisfy the condition of Theorem 3 than to satisfy the condition for the orthogonal array of strength 2. We need even more than 16 runs to satisfy both the condition of Theorem 3 and the condition for the orthogonal array of strength 2.

### 3.2 Balanced Arrays (B-arrays) of Full Strength.

We now consider a B-array  $T$  ( $n \times m$ ) of full strength with  $n$  distinct runs (see Srivastava and Chopra 1973). We first describe this B-array. Let  $S_i$  be the set of all  $(1 \times m)$  vectors having  $i$  elements equal to 1 and the other  $(m-i)$  elements equal to 0,  $i = 0, 1, \dots, m$ . The sets  $S_i$ ,  $i = 0, 1, \dots, m$ , appear as rows in  $T$  with frequencies  $\lambda_i$ , where  $\lambda_i = 0$  or 1,  $i = 0, 1, \dots, m$ . Clearly the number of vectors in  $S_i$  is  $\binom{m}{i}$  and  $n = \sum_{i=0}^m \lambda_i \binom{m}{i}$ .

Theorem 4. Consider a run  $\underline{t}' = (t_1, \dots, t_{u-1}, t_u, t_{u+1}, \dots, t_m)$  of a B-array of full strength. Suppose that 1 is appearing  $w$  times and 0 is appearing  $(m-w)$  times in  $\underline{t}'$ . Then  $\sigma^2(t_1, \dots, t_u = 1, \dots, t_m) = \sigma^2(\underline{t}'_{(1)})$  and  $\sigma^2(t_1, \dots, t_u = 0, \dots, t_m) = \sigma^2(\underline{t}'_{(0)})$  are both unbiasedly estimable if and only if the runs in  $S_w$  and  $S_{w+x}$ , where  $x = 1$  if  $t_u = 0$  and  $x = -1$  if  $t_u = 1$ , appear as rows in  $T$ .

Proof. Suppose  $\underline{t} = \underline{t}_{(1)}$ . Then the runs in  $S_w$  appear as rows in  $T$ . It now follows from the B-array of full strength that  $\underline{t}_{(0)}$  will appear as a row in  $T$  if and only if the runs in  $S_{w-1}$  appear as rows in  $T$ . The rest is clear. This completes the proof.

When  $S_w$  and  $S_{w+x}$ ,  $x = 1$  or  $-1$ , are present in  $T$ , we can estimate unbiasedly both  $\sigma^2(t_1^*, \dots, t_{u-1}^*, t_u^* = 1, t_{u+1}^*, \dots, t_m^*)$  and  $\sigma^2(t_1^*, \dots, t_{u-1}^*, t_u^* = 0, t_{u+1}^*, \dots, t_m^*)$  for every factor  $u (u = 1, \dots, m)$  and for some  $(t_1^*, \dots, t_{u-1}^*, t_{u+1}^*, \dots, t_m^*)$ . If the runs in  $T (n \times m)$ ,  $n = 1 + m$ , are the runs in  $S_0$  and  $S_1$  (or, equivalently,  $S_m$  and  $S_{m-1}$ ) then  $T$  is a resolution III plan and the condition of Theorem 4 holds. If the runs in  $T (n \times m)$ ,  $n = 1 + m + \binom{m}{2}$ , are the runs in  $S_0$ ,  $S_1$  and  $S_2$  (or, equivalently,  $S_m$ ,  $S_{m-1}$  and  $S_{m-2}$ ) then  $T$  is a resolution V plan and the condition of Theorem 4 holds for every factor. In view of the unbiased estimation of the dispersion effect  $\sigma^2(\underline{t}'_{(1)}) - \sigma^2(\underline{t}'_{(0)})$ , B-arrays of full strength are very good having smaller number of runs. The resolution IV plan where the runs in  $T (n \times m)$ ,  $n = 2 + 2m$ , are the runs in  $S_0$ ,  $S_m$ ,  $S_1$  and  $S_{m-1}$  is good in terms of estimating both location and dispersion effects. The resolution V plus one plan where the runs in  $T (n \times m)$ ,  $n = 2 + 2m + \binom{m}{2}$ ,

are the runs in  $S_0, S_m, S_1, S_{m-1}$  and  $S_2$  (see Srivastava and Ghosh (1976)) is good in terms of estimating both location and dispersion effects.

#### 4. Final Remarks

In the model (1-2) and throughout the paper, we present the dispersions in the general form. A special dispersion model assumes the additive form  $\sigma^2(t_1, \dots, t_m) = \sigma_1^2(t_1) + \dots + \sigma_m^2(t_m)$ . There are  $2m$  dispersion parameters  $\sigma_u^2(1)$  and  $\sigma_u^2(0)$ ,  $u = 1, \dots, m$  under the additive model. The additive form is implicitly assumed in the orthogonal array experimentations (see Taguchi and Wu 1985). The additivity in the orthogonal array experimentations simplifies the situations. For example, we have for the  $u$ th ( $u = 1, \dots, m$ ) factor,  $\sigma^2(1) - \sigma^2(0) = \sigma^2(\underline{t}'_1) - \sigma^2(\underline{t}'_0) = \sigma_u^2(1) - \sigma_u^2(0)$ . The strengths of the orthogonal arrays are assumed to be 2 or more. Under the additive model, the unbiased estimation of the dispersion effects  $\sigma_u^2(1) - \sigma_u^2(0)$ ,  $u = 1, \dots, m$ , is always possible in the orthogonal array experimentations.

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